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## A Simple Analysis of Reentry Vehicle Roll Resonance

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JANUARY 1967

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Prepared by D. H. PLATUS  
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Laboratories Division  
Laboratory Operations  
**AEROSPACE CORPORATION**

Prepared for **BALLISTIC SYSTEMS AND SPACE SYSTEMS DIVISIONS**  
**AIR FORCE SYSTEMS COMMAND**  
**LOS ANGELES AIR FORCE STATION**  
Los Angeles, California

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## FOREWORD

This report is published by the Aerospace Corporation, El Segundo, California, under Air Force Contract No. AF 04(695)-1001.

This report, which documents research carried out from November 1965 through December 1966, was submitted on 16 February 1967 to Captain John T. Allton, SSTRT, for review and approval.

Approved

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R. A. Hartunian, Acting Director  
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Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

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## ABSTRACT

Reentry vehicle roll resonance due to small mass and aerodynamic asymmetries is described analytically by a quasi-steady solution of the three-degree-of-freedom moment equations of motion. The quasi-steady analysis neglects oscillations in the vehicle orientation and angular rates about average (quasi-steady) values of these parameters, which change slowly with time relative to the oscillations. The study treats different regimes of reentry vehicle roll resonance that have been identified through computer solutions of the general equations of motion. These include high altitude roll lockin and breakout due to a single mass asymmetry, intermediate and low altitude subresonance with a single asymmetry, and spinup and lockin to low altitude resonance from a compound asymmetry. For each of the resonance regimes, analytical expressions are obtained that describe the quasi-steady vehicle motion, and the results are found to be in good agreement with computer solutions of the equations of motion. The analytical approximations provide a simple tool for predicting reentry vehicle dynamic behavior without requiring costly and time consuming machine computations.

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## NOMENCLATURE

c	center-of-mass asymmetry
$C_A$	aerodynamic axial force coefficient
$C_{I_p}$	aerodynamic roll damping derivative
$C_{m_q}$	aerodynamic pitch damping derivative
$C_{N\alpha}$	aerodynamic normal force derivative
$C_t$	aerodynamic trim force coefficient
d	aerodynamic reference diameter
$F_t$	trim force
h	angular momentum; altitude
H	reference altitude for exponential atmosphere
I	pitch or yaw moment of inertia
$I_x$	roll moment of inertia
$\lambda$	static margin (distance of center of pressure aft of center of mass)
$L_t$	location of trim force aft of center of mass
m	vehicle mass
M	moment
p	roll rate
$p_c$	critical roll rate, $(\frac{\lambda}{1 - \mu})^{1/2}$
q	dynamic pressure
S	aerodynamic reference area
t	time
u	vehicle velocity

$x, y, z$	body-fixed axes
$X, Y, Z$	inertial axes
$\beta$	ballistic coefficient, $\frac{mg}{C_A S}$
$\gamma$	path angle
$\delta$	$\frac{C_A q Sc}{I}$
$\epsilon$	roll damping coefficient, $- \frac{C_I q S d^2}{I_x U}$
$\zeta$	roll axis
$\eta$	yaw axis
$\theta$	total angle of attack
$\kappa$	roll moment coefficient, $\frac{C_N q S c}{I_x}$
$\lambda$	pitch moment coefficient, $\frac{C_N q S l}{I}$
$\mu$	inertia ratio, $I_x/I$
$\nu$	pitch damping coefficient, $\frac{q S d^2}{I_u} \left[ - C_{m_q} + \frac{C_N I}{m d^2} \right]$
$\xi$	pitch axis
$\rho$	atmospheric density
$\sigma$	trim roll moment coefficient, $\frac{C_t q S c}{I_x}$
$\tau$	trim pitch moment coefficient, $\frac{C_t q S L_t}{I}$

$\phi$  roll orientation relative to wind

$\psi$  precession angle

$\omega$  angular velocity

Subscripts

$\zeta$  roll

$\eta$  yaw

$\xi$  pitch

$+$  oscillatory component

Superscripts

$\cdot$  time derivative

$-$  quasi-steady component

## I. INTRODUCTION

It has been known for some time that missiles or projectiles with slight configurational asymmetries can exhibit erratic roll behavior due to gyroscopic coupling of roll and pitch motions (roll resonance) at or near the natural pitch frequency of the missile (Refs. 1 and 2). More recently, with the advent of smaller and higher ballistic coefficient reentry vehicles, similar erratic behavior has been observed that can have adverse and even catastrophic effects on the vehicle motion during reentry (Refs. 3 and 4).

Because of the complexity of the equations of motion when gyroscopic coupling is involved, there are few analytical treatments of roll resonance, and much of what is known has been obtained from computer solutions of the general equations of motion. Such studies provide detailed information on the motions and trajectories of specific vehicles but, unfortunately, do not lend themselves to identifying in a scientific manner the pertinent parameters that contribute to the roll resonance phenomena. The purpose of the present study is to identify analytically the most important of these parameters through a solution of a simple set of equations embodying the essential features of reentry vehicle dynamics. The results so obtained provide a simple tool for predicting the occurrence and severity of reentry vehicle roll resonance as a function of vehicle configuration, aerodynamic, and trajectory parameters.

The study treats, separately, three regimes of resonance identified in Ref. 4, in the order in which they might be encountered by a typical reentry vehicle. First, the high altitude resonance when the vehicle roll rate first interacts with the critical roll rate is examined, and the conditions necessary for roll lockin and breakout due to a single mass asymmetry are described. Next, the steady subresonant condition exhibited by a vehicle with a single mass asymmetry when the roll rate lies below the critical roll rate is described analytically, and the results are compared with computer solutions of the three-degree-of-freedom equations of motion. Finally, an approximate

method is presented for predicting the transient roll spinup to resonance and concomitant angle-of-attack divergence induced by a compound asymmetry, and these results are also compared with computer solutions of the equations of motion.

## II. COORDINATES AND EQUATIONS OF MOTION

It is convenient for the purpose of analysis to consider only three-degree-of-freedom rotational motion, since the roll resonance phenomenon is basically gyroscopic in nature. There is, however, a normal force contribution to pitch or yaw damping from lateral motion of the center of mass, which can be accounted for in specifying the aerodynamic moments, so that the three moment equations effectively account for five-degree-of-freedom motion for an axisymmetric vehicle.

The rotational motion is described in terms of the Euler angles  $\psi$ ,  $\phi$ ,  $\theta$ , as shown in Figure 1, which describe the position of the body-fixed axes  $x$ ,  $y$ ,  $z$  relative to a set of inertial axes  $X$ ,  $Y$ ,  $Z$ . It is assumed that the inertial frame of reference is moving with the vehicle center of mass along the velocity vector, which coincides with the direction of the  $X$ -axis, and that the vehicle axis is the body-fixed axis  $x$  or  $\zeta$ . The angles  $\psi$ ,  $\phi$ ,  $\theta$ , which are rotations about the axes  $X$ ,  $x$ ,  $\zeta$ , respectively, then represent precession about the velocity vector, roll orientation relative to the wind, and total angle of attack, respectively. The mutually perpendicular axes  $\xi$ ,  $\eta$ ,  $\zeta$ , which precess about the velocity vector, are the pitch, yaw, and roll axes, respectively, where the pitch plane is defined as the plane of total angle of attack. These axes differ from the usual body-fixed axes  $x$ ,  $y$ ,  $z$  in that the pitch axis  $\xi$  (the line of nodes in the Eulerian system) is always perpendicular to the direction  $X$  of the velocity vector, thereby defining the plane of the total angle of attack, whereas the body-fixed axes  $y$  and  $z$  rotate in this system about the roll axis ( $x$  or  $\zeta$ ) with angular rate  $\dot{\phi}$ .

If the principal moments of inertia about the  $\xi$ ,  $\eta$ ,  $\zeta$  axes are  $I_x$ ,  $I_y$ ,  $I_z$ , respectively (pitch, yaw, roll), the angular velocities of the  $\xi$ ,  $\eta$ ,  $\zeta$  axes and the angular momentum about them are (Ref. 5)

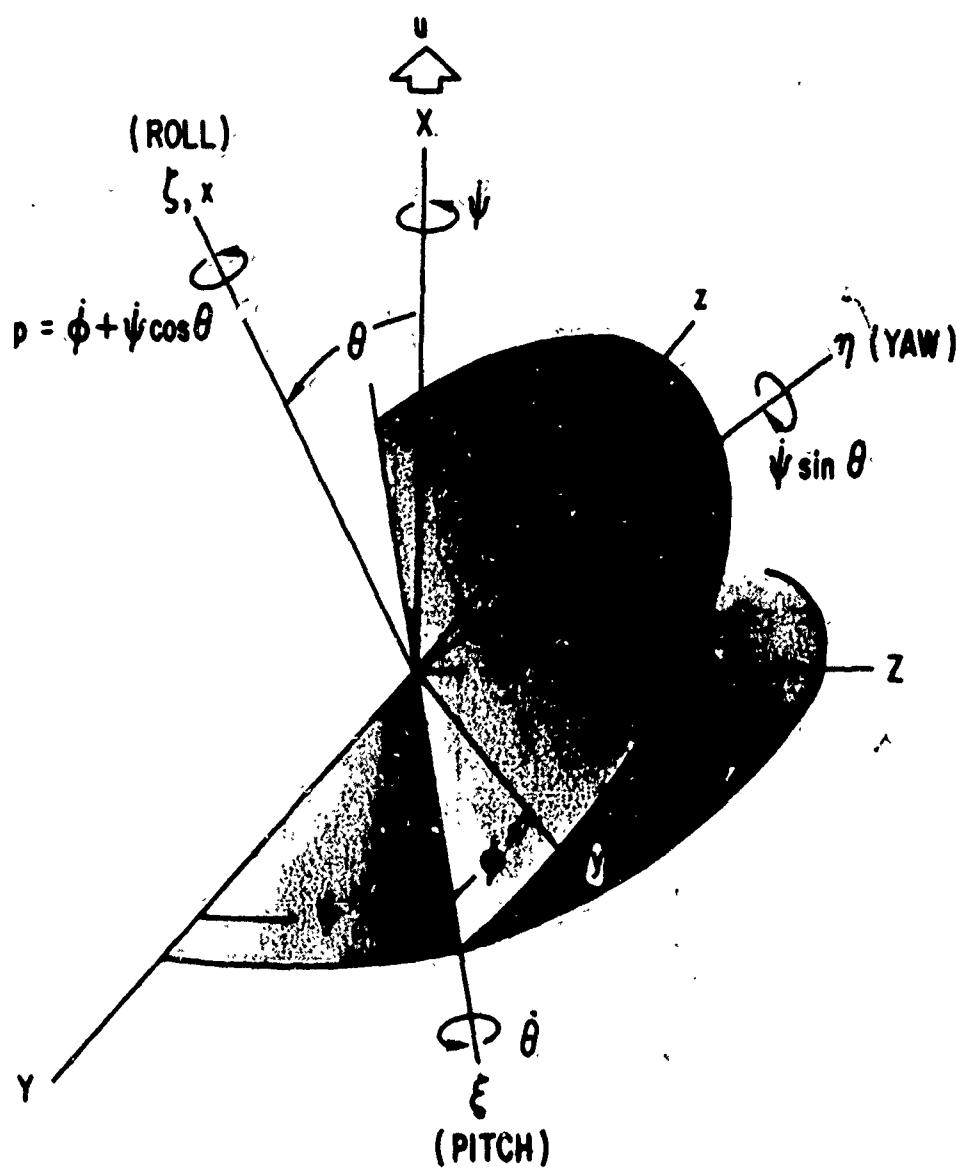


Figure 1. Euler's Angles for Three-Degree-of-Freedom Rotational Motion

$$\left. \begin{array}{l} \omega_\xi = \dot{\theta} \quad h_\xi = I\ddot{\theta} \\ \omega_\eta = \dot{\psi} \sin \theta \quad h_\eta = I\dot{\psi} \sin \theta \\ \omega_\zeta = \dot{\psi} \cos \theta \quad h_\zeta = I_x(\dot{\phi} + \dot{\psi} \cos \theta) \end{array} \right\} \quad (1)$$

where  $\omega_\xi$  and  $\omega_\eta$  are the pitch and yaw rates, respectively, and the roll rate is the sum

$$p = \dot{\phi} + \dot{\psi} \cos \theta \quad (2)$$

The moment equation of motion is

$$\vec{M} = \dot{\vec{h}} + \vec{\omega} \times \vec{h} \quad , \quad (3)$$

which, from Eq. (1) with the definition, Eq. (2), has the components

$$\left. \begin{array}{l} M_\xi = I\ddot{\theta} + I_x p \dot{\psi} \sin \theta - I\dot{\psi}^2 \sin \theta \cos \theta \\ M_\eta = I \frac{d}{dt} (\dot{\psi} \sin \theta) + I\dot{\theta} \dot{\psi} \cos \theta - I_x p \dot{\theta} \\ M_\zeta = I_x \frac{dp}{dt} \end{array} \right\} \quad (4)$$

The moments  $M_\xi$ ,  $M_\eta$ ,  $M_\zeta$  are the aerodynamic pitch, yaw, and roll moments, respectively, acting on the vehicle.

### III. SINGLE MASS ASYMMETRY

#### A. MOMENT EQUATIONS

It will be assumed initially that the only asymmetry that gives rise to pitch roll coupling is a displacement of the vehicle center of mass from the axis of symmetry (center of pressure) of the vehicle and that this displacement is small enough that the resulting cross products of inertia can be ignored. Denoting this displacement  $c$ , as shown in Figure 2, we can write the aerodynamic moments in terms of normal and axial force components for an otherwise axially symmetric vehicle<sup>1</sup>

$$\left. \begin{aligned} M_\xi &= C_{N\alpha} q S l \theta + C_A q S c \cos \phi + \frac{q S d^2}{u} \left[ C_{m_q} - \frac{C_{N\alpha} I}{m d^2} \right] \dot{\theta} \\ M_\eta &= C_A q S c \sin \phi + \frac{q S d^2}{u} \left[ C_{m_q} - \frac{C_{N\alpha} I}{m d^2} \right] \dot{\psi} \sin \theta \\ M_\zeta &= -C_{N\alpha} q S \theta c \sin \phi + \frac{C_I q S d^2 p}{u} \end{aligned} \right\} \quad (5)$$

<sup>1</sup>The term  $C_{N\alpha} I / m d^2$  in the first two of Eqs. (5) is the usual normal force damping term from lateral motion of the vehicle center of mass. It can be shown for an axisymmetric vehicle that the lateral motions are proportional to pitch and yaw rotations about a point a distance  $I/m l$  ahead of the center of mass. Thus the three resulting moment equations account for lateral translation of the vehicle and effectively describe five-degree-of-freedom motion.

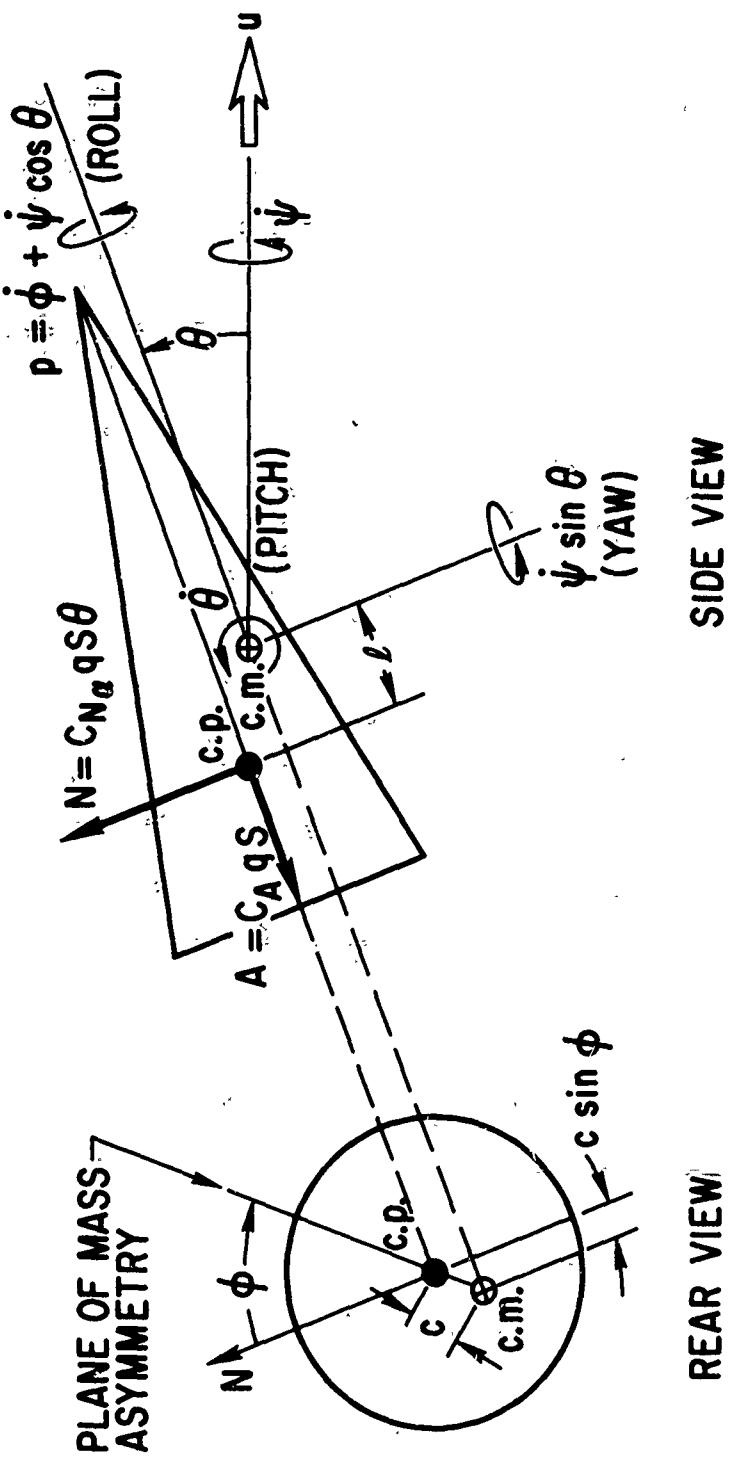


Figure 2. Aerodynamic Force Diagram

where  $C_{mq}$  and  $C_{lp}$  are pitch and roll damping derivatives, respectively, and the angle  $\phi$  describing the roll orientation relative to the wind is zero in the statically stable position when the center of pressure is on the leeward side of the center of mass. Negative  $\phi$  produces a positive roll torque and positive  $\phi$  produces a negative roll torque. It is shown in Section III-B that roll resonance is a condition during which  $\phi$  oscillates about a nonzero value of  $\phi$  producing a net roll torque. The magnitude and direction of this torque are strongly influenced by the magnitude and time rate of change of the dynamic pressure. The oscillation in  $\phi$  about a steady value or one slowly varying in time is defined here as lunar motion in that a certain meridian of the vehicle is preferentially facing into the wind. From this definition, as is shown in Section III-B, lunar motion is a necessary condition for resonance.

Substituting Eqs. (5) in Eqs. (4) and dividing the first two equations by  $I$  and the third by  $I_x$ , we can write the equations of motion

$$\left. \begin{aligned} -\lambda \theta + \delta \cos \phi - \nu \dot{\theta} &= \ddot{\theta} + \mu p \dot{\psi} \sin \theta - \dot{\psi}^2 \sin \theta \cos \theta \\ \delta \sin \phi - \nu \dot{\psi} \sin \theta &= \frac{d}{dt} (\dot{\psi} \sin \theta) + \dot{\theta} \dot{\psi} \cos \theta - \mu p \dot{\theta} \\ -\kappa \theta \sin \phi - \epsilon p &= \frac{dp}{dt} \end{aligned} \right\} \quad (6)$$

where the coefficients are

$$\left. \begin{aligned} \lambda &= \frac{C_N q S l}{I} & \kappa &= \frac{C_N q S c}{I_x} \\ \delta &= \frac{C_A q S c}{I} & \epsilon &= -\frac{C_l q S d^2}{I_x u} \\ \nu &= \frac{q S d^2}{I u} \left( -C_{mq} + \frac{C_N I}{m d^2} \right) & \mu &= \frac{I_x}{I} \end{aligned} \right\} \quad (7)$$

The sign convention is such that for a statically stable vehicle with positive damping ( $C_{m_q} < 0$ ,  $C_{l_p} < 0$ ), all of the coefficients will be positive.

B. QUASI-STEADY SOLUTION FOR HIGH ALTITUDE  
ROLL LOCKIN AND BREAKOUT

It will be assumed that the vehicle motion during reentry, described by  $[\psi(t), \phi(t), \theta(t)]$ , is of the form

$$[\psi(t), \phi(t), \theta(t)] = [\bar{\psi}(t), \bar{\phi}(t), \bar{\theta}(t)] + [\psi_+(t), \phi_+(t), \theta_+(t)] , \quad (8)$$

where  $[\bar{\psi}(t), \bar{\phi}(t), \bar{\theta}(t)]$  represents a quasi-steady component that varies relatively slowly with time (of the order of the dynamic pressure) and  $[\psi_+(t), \phi_+(t), \theta_+(t)]$  represents an oscillation of higher frequency about the quasi-steady values. It is of interest to examine solutions to the equations of motion for the quasi-steady components only.

Making the small angle approximations  $\sin \theta \approx \theta$ ,  $\cos \theta \approx 1$ , neglecting pitch damping  $\nu$ , and ignoring the term  $\delta \cos \phi$ , which is quite small except at angles of attack near static trim, we can write the first of Eqs. (6) in the form

$$\ddot{\theta} + (\lambda + \mu p \dot{\psi} - \dot{\psi}^2) \theta = 0 \quad . \quad (9)$$

This equation describes a nonlinear oscillation in  $\theta$  about a nonzero quasi-steady value  $\bar{\theta}$ , since  $\theta$ , by definition, is always positive. The quasi-steady values  $\bar{\psi}$  and  $\bar{p}$  corresponding to  $\bar{\theta}$  are those satisfying the equation

$$\lambda + \mu \bar{p} \bar{\dot{\psi}} - \bar{\dot{\psi}}^2 = 0 \quad , \quad (10)$$

which makes  $\ddot{\theta}$  in Eq. (9) zero. Since, in general,  $\mu \ll 1$ , the precession rate  $\dot{\psi}$  from Eq. (10) is approximately equal to  $\pm \lambda^{1/2}$ , the pitch natural frequency of the vehicle. From the definition, Eq. (7),  $\lambda^{1/2}$  is proportional to the square root of the dynamic pressure and is a relatively slowly varying function of time.

Similarly, we can write the third of Eqs. (6), using the definition of  $p$  from Eq. (2) with the assumption  $\cos \theta \approx 1$ , in the form

$$\ddot{\phi} + \kappa \theta \sin \phi = \epsilon p - \frac{d}{dt} \bar{\psi}, \quad (11)$$

which, under certain conditions when the right hand side is a slowly varying function of time, represents a nonlinear oscillation in  $\phi$  about the nonzero quasi-steady value  $\bar{\phi}$  satisfying the equation

$$\sin \bar{\phi} = \frac{1}{\kappa \theta} \left( \epsilon \bar{p} - \frac{d}{dt} \bar{\psi} \right). \quad (12)$$

From Eqs. (2) and (10) with the assumption of small  $\theta$ , the existence of a quasi-steady value  $\bar{\phi}$  such that  $d\bar{\phi}/dt \approx 0$  requires that

$$\bar{p} \approx \bar{\psi} \approx \pm \left( \frac{\lambda}{1 - \mu} \right)^{1/2} \equiv \pm p_c \quad (13)$$

Eq. (13) is the lunar motion condition of roll resonance during which the roll and precession rates are equal to the critical roll rate  $p_c$ , which is slightly greater than the natural pitch frequency  $\lambda^{1/2}$  of the vehicle. The resonance condition can exist only when  $\sin \bar{\phi}$  given by

$$\sin \bar{\phi} = \frac{1}{\kappa \theta} \left( \pm \epsilon p_c \mp \frac{dp_c}{dt} \right) \quad (14)$$

is real, i.e., when

$$|\sin \bar{\phi}| \leq 1 \quad (15)$$

Hence, the criterion, Eq. (15), with  $\bar{\phi}$  defined by Eq. (14), is a necessary condition for resonance for a vehicle having a single mass asymmetry. Rewriting  $\kappa$  in Eq. (14) in terms of  $p_c$  using the definitions, Eqs. (13) and (7), we can write  $\sin \bar{\phi}$  in the form

$$\sin \bar{\phi} = \frac{(l/c) I_x / (I - I_x)}{p_c^2 \theta} \left( \pm \epsilon p_c \mp \frac{dp_c}{dt} \right) \quad (16)$$

Since roll damping  $\epsilon$  is generally quite small,  $\sin \bar{\phi}$  from Eq. (16) is approximately proportional to the slope of the critical roll rate and is inversely proportional to the square of the critical roll rate and angle of attack.

If the vehicle has an initial exoatmospheric roll rate, say  $p_0$ , then at some point in the trajectory, say  $t_1$ , the vehicle roll rate must intersect the critical roll rate curve, as shown in Figure 3.

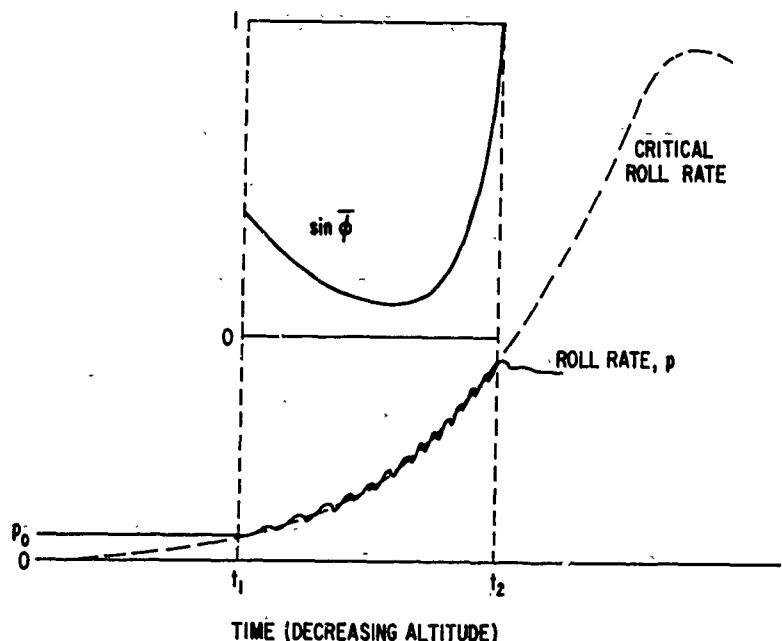


Figure 3. High Altitude Roll Resonance

At this point the vehicle will lock into resonance, provided that the criterion, Eq. (15), is satisfied and that the vehicle has the proper roll orientation with respect to the wind; i.e., the angle  $\phi$  must be such that the induced roll torque has the same sense as the critical roll acceleration (e.g., if  $\dot{p}_c$  is positive, then  $\phi$  as defined in Figure 2 must be negative to produce a positive roll torque). Hence, the  $\sin \phi$  criterion, Eq. (15), is a necessary, but not sufficient, condition for resonance in that lockin depends also on the phasing of  $\phi$  as the roll rate approaches critical. However, if lockin does occur, then the vehicle must break out of resonance when  $\sin \phi$  is no longer real, which will generally occur at some later point, such as  $t_2$  in Figure 3, when  $\dot{p}_c$  is large and  $\theta$  has damped to a sufficiently small value. The breakout point has particular physical significance. Referring to Figure 2 and the third of Eqs. 6, the roll torque which produces the roll acceleration  $\dot{p}$  is proportional to the product  $\bar{\theta} \sin \phi$ . When  $\bar{\theta}$  becomes sufficiently small, the maximum possible roll torque, corresponding to  $\phi = 90$  deg, is insufficient to increase  $\dot{p}$  at an average rate equal to the time rate of change of the critical roll rate, and resonance is no longer possible.

In order to evaluate  $\sin \phi$  from Eq. (16), we require a knowledge of the angle-of-attack convergence and the dynamic pressure history, which determines the magnitude and slope of the critical roll rate. The dynamic pressure is approximated assuming an exponential atmosphere with constant path angle, the details of which are discussed in Appendix A. The angle-of-attack convergence is obtained from a quasi-steady solution of the equations of motion, which is summarized in Appendix B. Using these results, we can now express the  $\sin \phi$  criterion, Eq. (16), in terms of the vehicle and trajectory parameters.

At  $t = t_1$  in Figure 3, where the roll rate first intersects the critical curve, the roll rate is approximately equal to the exoatmospheric value  $p_\infty$  and the angle of attack has the approximate value given by Eq. (B-13), Appendix B. Ignoring roll damping  $\epsilon$  and substituting  $\sin \bar{\theta}_1 \approx \bar{\theta}_1$  from Eq. (B-13) with the slope  $dp_c/dt$  from Eq. (A-16), Appendix A, in Eq. (16), we obtain for  $\sin \phi$ ,

$$|\sin \bar{\phi}_1| = \left( \frac{\mu}{1-\mu} \right) \left( \frac{I}{c} \right) \frac{gu_0(p_* - p_1)}{2\beta p_0} \left( \frac{\cos \theta_0}{\mu} \right)^{1/2} \frac{\exp[(1/2)(b-a)p_1 - (1/2)bp_0]}{\sin \theta_0}, \quad (17)$$

which, depending on whether or not  $\bar{\phi}_1$  is real, i.e.,  $|\sin \bar{\phi}_1| \leq 1$ , is an approximate criterion for determining roll lockin. Assuming that lockin does occur, the breakout point can also be estimated using Eq. (16).

Equation (B-15), Appendix B, describes the angle-of-attack convergence while the vehicle is in resonance. Substituting this expression for  $\bar{\theta}$  in Eq. (16), using the values for  $p_c$  and  $dp_c/dt$  from Appendix A, as before, we obtain for the  $\sin \bar{\phi}$  expression during lockin

$$|\sin \bar{\phi}| = \frac{u_0^{1/2}}{2^{3/4}} \left( \frac{I}{c} \right) \left( \frac{g}{\beta} \right) \left( \frac{\mu}{1-\mu} \right) \left( \frac{I - I_x}{C_N \cdot S_l} \right)^{1/4} \left( \frac{p_* - p}{p^{1/4}} \right) \frac{\exp[(b/2)(p - p_1) - (a/4)p]}{[B - \delta(p)]^{1/2}}, \quad (18)$$

where

$$\begin{aligned} \delta(p) \equiv & \frac{2Ae^{\alpha p}}{a\alpha^{1/2}} \left\{ \frac{b}{\alpha} \left[ D(\sqrt{\alpha p}) - e^{-\alpha(p-p_1)} D(\sqrt{\alpha p_1}) \right] \right. \\ & \left. - \frac{ap^{1/2}}{2\alpha^{1/2}} \left[ 1 - \left( \frac{p_1}{p} \right)^{1/2} e^{-\alpha(p-p_1)} \right] \right\} \end{aligned} \quad (19)$$

$$\left. \begin{aligned} a &\equiv \frac{gH}{\beta \sin \gamma} \\ b &\equiv \frac{HS}{2 \sin \gamma} \left[ -\frac{C_m d^2}{I} + \frac{C_{N_\alpha}}{m} \right] \\ \alpha &\equiv b - \frac{a}{2} \\ A &\equiv \frac{\mu C_A}{C_{N_\alpha}} \left( \frac{C_{N_\alpha} S_l}{I - I_x} \right)^{1/2} \frac{gu_0 H}{2\sqrt{2}\beta \sin \gamma} e^{-bp_1} \end{aligned} \right\} \quad (20)$$

and  $D(x)$  is Dawson's integral defined by

$$D(x) \equiv e^{-x^2} \int_0^x e^{t^2} dt ,$$

for which tabular values are available (see, for example, Ref. 6). Breakout must occur when  $|\sin \bar{\phi}|$  from Eq. (18) approaches unity, as shown in Figure 3.

Figure 4 shows a plot of  $\sin \bar{\phi}$  from Eq. (18) for a typical reentry vehicle, for different values of damping, measured from the point at which the roll rate first intersects the critical curve,  $p_o = p_c$ . Figure 5 shows a comparison of roll rates computed from a numerical integration of Eqs. (6), with the estimated breakout points based on the  $\sin \bar{\phi}$  criterion of Eq. (15), for the values plotted in Figure 4. Also shown in Figure 5 is a comparison of the computed angle-of-attack history with the quasi-steady approximation of Appendix B for  $C_{mq} = -4$ . It is seen that for greater damping, breakout occurs earlier. This is a consequence of the increased angle-of-attack convergence, which causes  $|\sin \bar{\phi}|$  to increase more rapidly to its limiting value of unity. It is also seen that the greater the damping, the more accurately the breakout point is predicted. This is a consequence of the assumption of quasi-steady motion. It was shown that the vehicle breaks out of resonance when the maximum roll moment corresponding to  $\bar{\phi} = 90$  deg is insufficient to accelerate the roll at the critical roll acceleration. Since the average moment is computed based on the average moment arm  $c \sin \bar{\phi}$  (Figure 2), the moment will be overestimated, depending on the amplitude of the  $\phi$  oscillation, since the average value of  $\sin \phi$  at  $\bar{\phi} = 90$  deg is less than unity; i.e.,  $\sin \bar{\phi} < \sin \phi$ . The lower the damping, the greater the amplitude of oscillation will be, which accounts for the discrepancies of Figure 5.

### C. SUBRESONANT CONDITION

Following breakout, which is accompanied by a perturbation in the vehicle orientation and angular rates, and in the absence of other asymmetries,

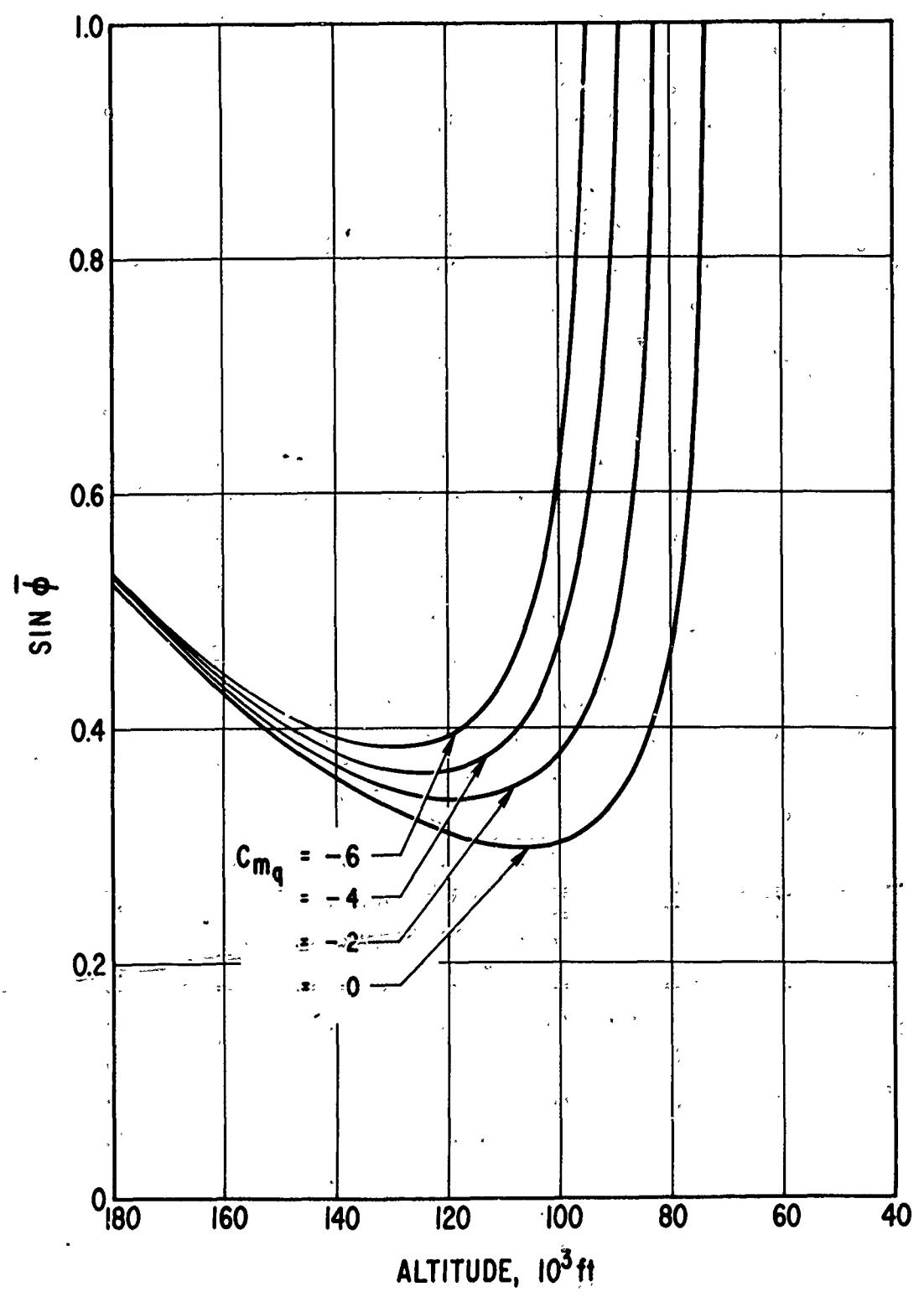


Figure 4.  $\sin \phi$  vs Altitude for Typical Reentry Vehicle

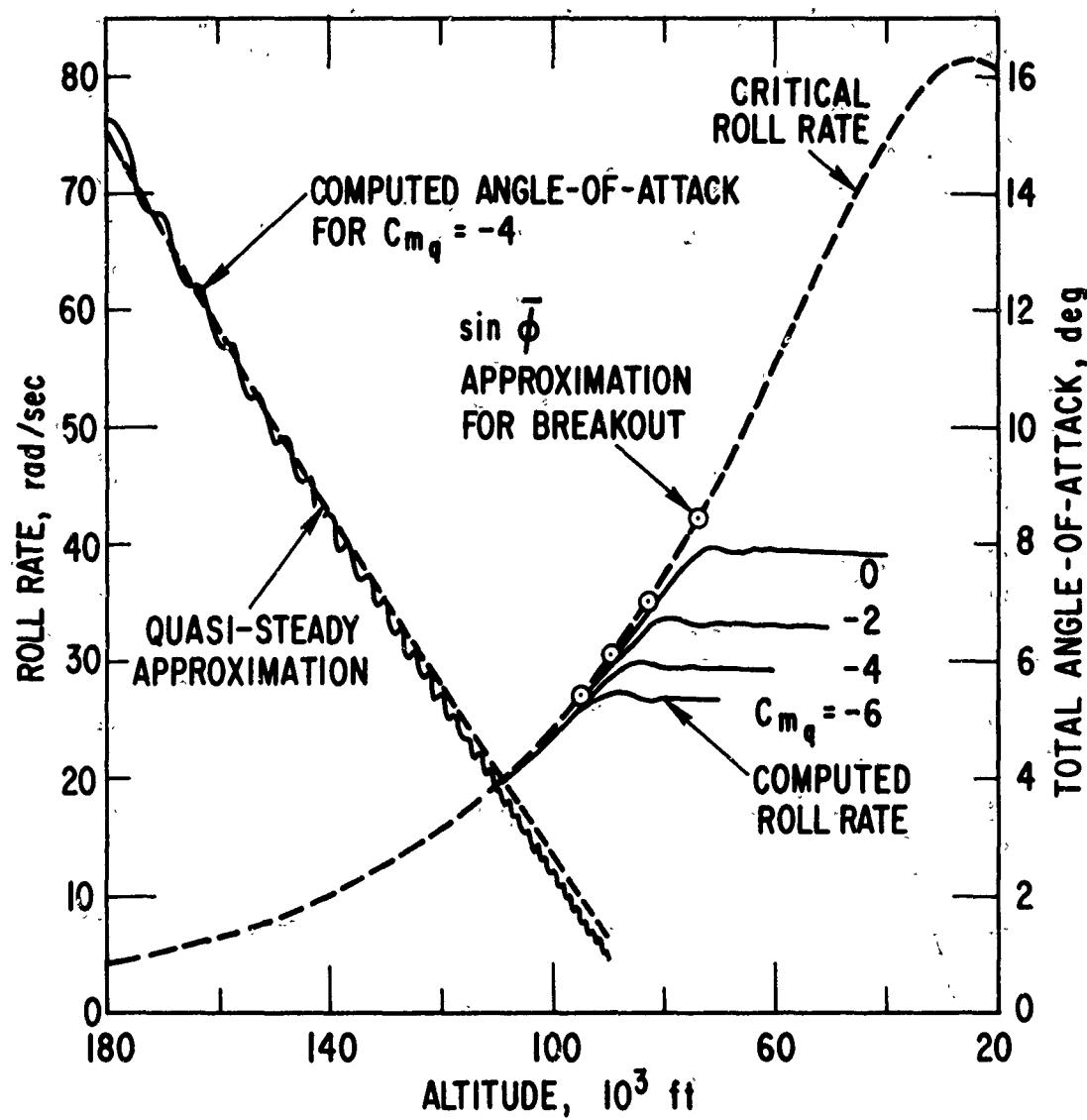


Figure 5. Comparison of Quasi-Steady Approximation and Computer Calculation of Breakout Roll Rate and Angle-of-Attack Convergence during High Altitude Resonance

the roll rate has been observed to become almost constant, decreasing slightly, and a condition of steady lunar motion exists at very small angle of attack. This is a stable subresonant condition that, owing to its steady nature, can be described quite accurately by a quasi-steady solution of the equations of motion.

Assuming that the angle of attack and angular rates remain constant, i.e.,  $\dot{\theta} \approx 0$  and  $\ddot{\theta} \approx \ddot{\psi} \approx \bar{n} \approx 0$ , and the small angle approximations are valid, we can write the first two of Eqs. (6) in the form

$$\left. \begin{aligned} \delta \cos \bar{\phi} - (\lambda + \mu \bar{\psi} - \bar{\psi}^2) \bar{\theta} &= 0 \\ \delta \sin \bar{\phi} - \nu \bar{\psi} \bar{\theta} &\approx 0 \end{aligned} \right\}, \quad (21)$$

where the bars, as before, denote quasi-steady values. Since  $\bar{\phi}$  from Eqs. (21) must also be slowly varying, which is the lunar motion condition  $d/dt \bar{\phi} \approx 0$  or  $\bar{n} \approx \bar{\psi}$ , Eqs. (21) may be written

$$\tan \bar{\phi} = \frac{\nu p_*}{\lambda \left[ 1 - (p_*/p_c)^2 \right]} \quad (22)$$

and

$$\bar{\theta} = \frac{\delta \cos \bar{\phi}}{\lambda \left[ 1 - (p_*/p_c)^2 \right]}, \quad (23)$$

where  $p_*$  represents the almost constant value of  $p$  after breakout. For  $p_*$  sufficiently removed from  $p_c$ , as in Figure 5,  $\tan \bar{\phi} \approx \bar{\phi} \ll 1$ ,  $\cos \bar{\phi} \approx 1$ , and we can write Eq. (23) in the form

$$\frac{\bar{\theta}}{\theta_{\text{trim}}} = 1 - (p_*/p_c)^2, \quad (24)$$

where  $\theta_{\text{trim}} = \delta/\lambda$  is the static or nonrolling trim angle of attack due to the mass asymmetry. Thus, it is seen that this trim is amplified by the factor  $[1 - (p_*/p_c)^2]^{-1}$ , which is characteristic of a near-resonant condition. It is shown in Section IV-B that an aerodynamic trim in conjunction with the mass asymmetry, which can cause spinup into low altitude resonance, is amplified by this same factor during spinup.

The slight roll deceleration during the subresonant condition may be estimated from the third of Eqs. (6) upon substitution of  $\sin \bar{\phi} \approx \tan \bar{\phi}$  from Eq. (22) and  $\bar{\theta}$  from Eq. (23), which gives

$$\dot{p}_* = - \frac{\kappa \delta v p_*}{\lambda^2 [1 - (p_*/p_c)^2]^2} \quad (25)$$

This deceleration is found to be quite small except as  $p_* \rightarrow p_c$ .

Figure 6 shows a comparison of the quasi-steady angle of attack from Eq. (24) with that obtained from a computer solution of the equations of motion, for the case  $C_{m_q} = -4$  discussed previously. Also shown in the figure are the computed roll rate and quasi-steady approximation based on the breakout roll rate from Figure 5 with a numerical integration of Eq. (25).

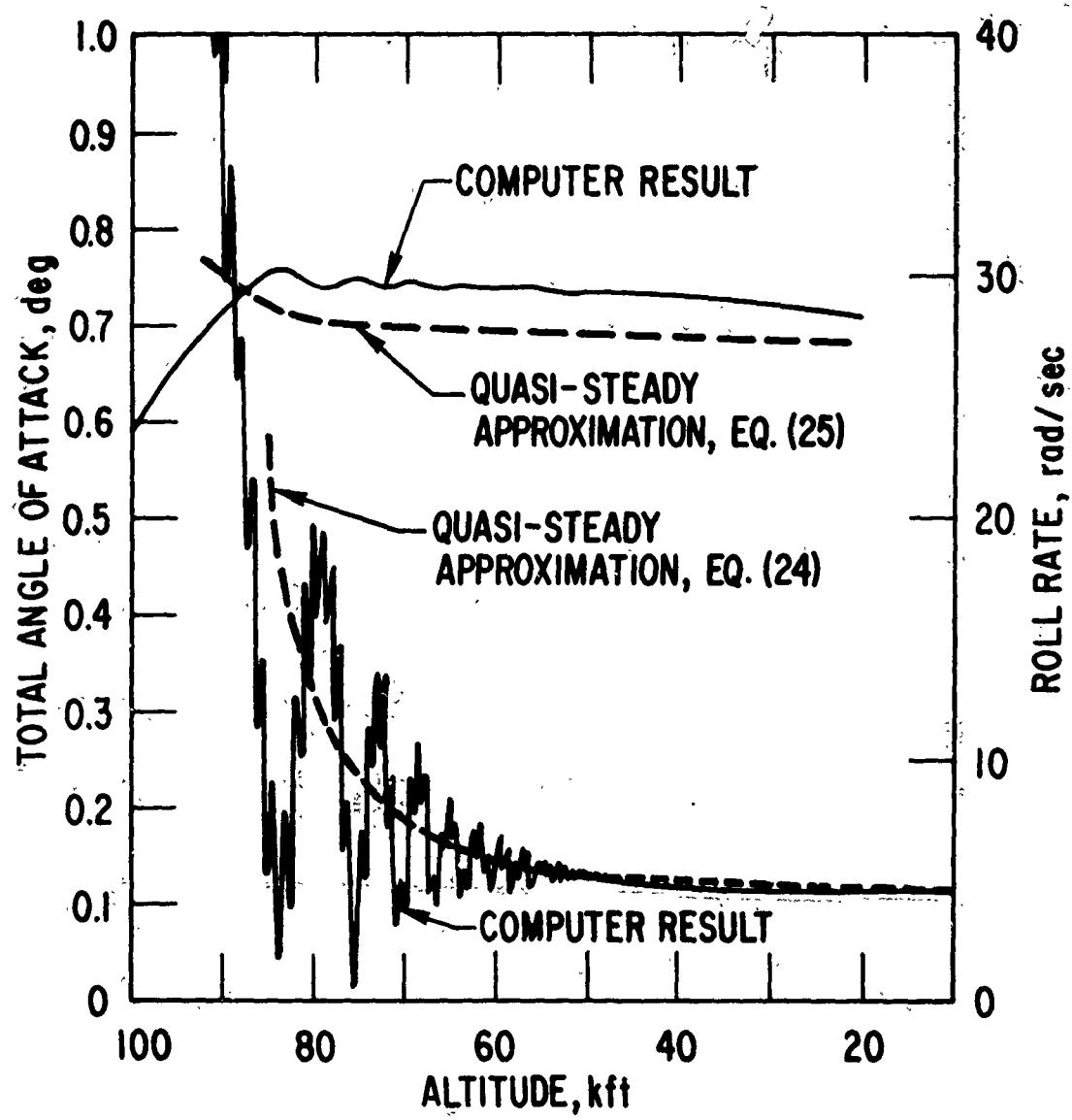


Figure 6. Comparison of Quasi-Steady Approximation and Computer Calculation of Angle-of-Attack and Roll Rate during Subresonance

## IV. COMPOUND ASYMMETRY

### A. MOMENT EQUATIONS WITH AERODYNAMIC TRIM

It has been shown (Ref. 4) that if a relatively small aerodynamic trim force develops at lower altitude that acts out of the plane of the center-of-mass asymmetry, a roll acceleration will be produced that can be sufficient to spin the vehicle into low altitude resonance. This trim can be represented by the normal force  $F_t$  oriented at an angle  $\phi_0$  with respect to the plane of the center-of-mass asymmetry, as shown in Figure 7, and acting at a distance  $L_t$  aft (or fore) of the center of mass. Referring to Figure 2 and defining a trim force coefficient  $C_t = F_t/qS$ , we write the resulting pitch, yaw, and roll moments in the form

$$\left. \begin{aligned} M_\xi &= C_t q S L_t \cos(\phi + \phi_0) \\ M_\eta &= C_t q S L_t \sin(\phi + \phi_0) \\ M_\zeta &= -C_t q S c \sin \phi_0 \end{aligned} \right\} \quad (26)$$

When these terms are added to the moments of Eq. (5), the equations of motion, Eqs. (6), become

$$\left. \begin{aligned} \tau \cos(\phi + \phi_0) - \lambda \theta + \delta \cos \phi - v \dot{\theta} &= \ddot{\theta} + \mu p \dot{\psi} \sin \theta - \dot{\psi}^2 \sin \theta \cos \theta \\ \tau \sin(\phi + \phi_0) - v \dot{\psi} \sin \theta &= \frac{d}{dt}(\dot{\psi} \sin \theta) + \dot{\theta} \dot{\psi} \cos \theta - \mu p \dot{\theta} \\ -\sigma \sin \phi_0 - \kappa \theta \sin \phi - \epsilon p &= \frac{dp}{dt} \end{aligned} \right\} \quad (27)$$

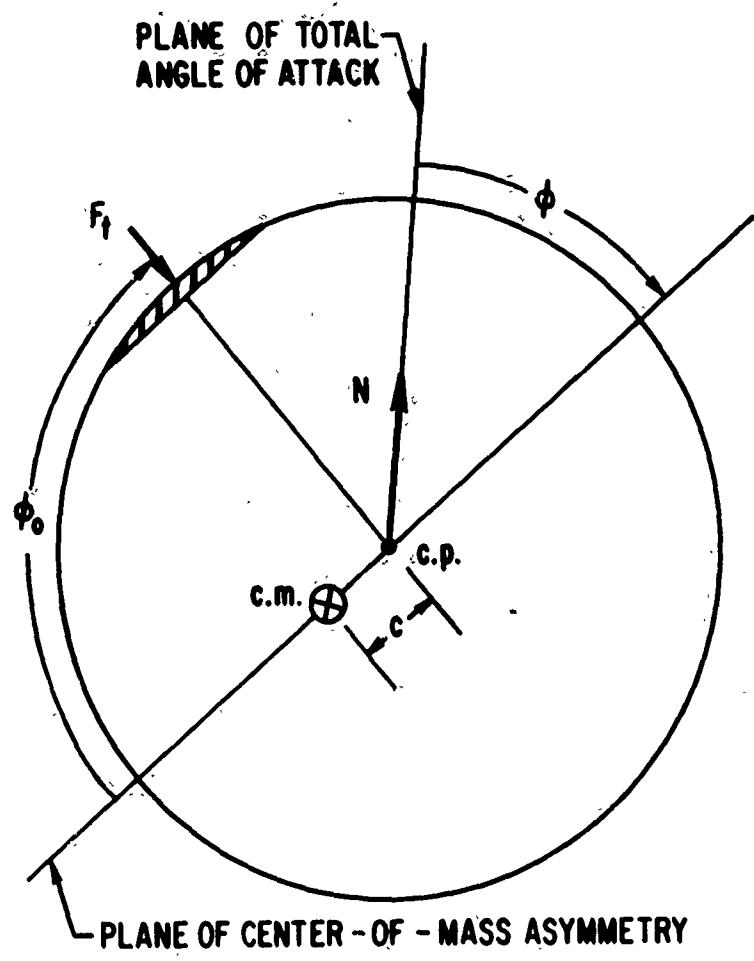


Figure 7. Orientation of Aerodynamic Trim Force

where  $\tau$  and  $\sigma$  are defined by

$$\tau \equiv \frac{C_t q S L_t}{I}, \quad \sigma \equiv \frac{C_t q S c}{I_x} \quad (28)$$

When  $\phi_0 \approx \pm 90$  deg, a relatively small normal trim force acting at a sufficient distance  $L_t$  from the center of mass (at the fore or aft ends of the vehicle, for example) will cause the roll to accelerate from its relatively constant subresonant value to the critical value, as shown in Figure 8. On reaching the critical roll rate the vehicle will once again lock into resonance with a subsequent divergence in angle of attack and increase in lateral loading. The same trim introduced at higher altitude prior to breakout from resonance would cause the vehicle to remain locked into resonance.

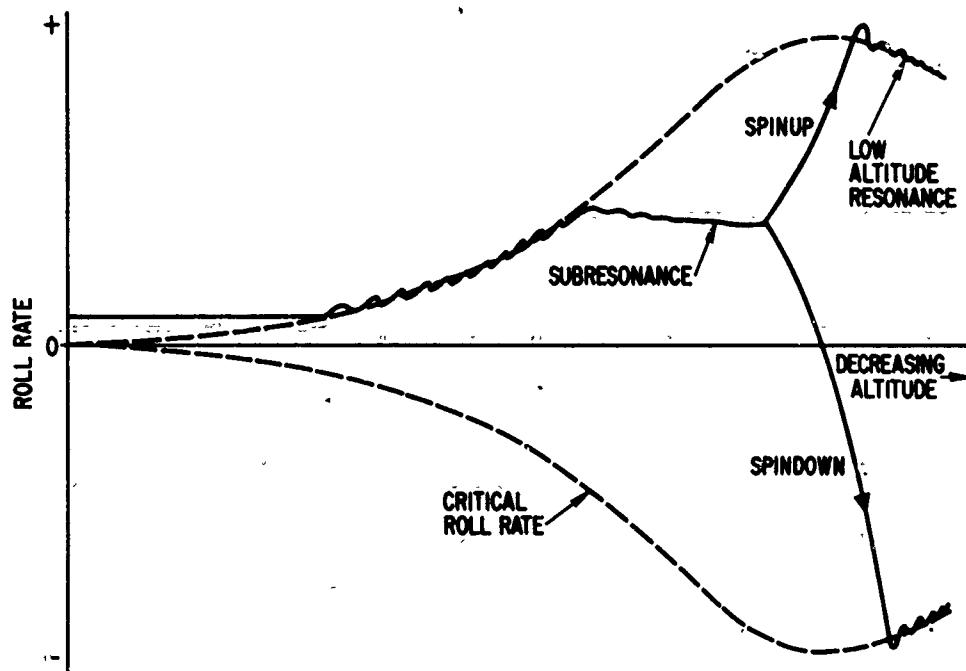


Figure 8. Spinup to Low Altitude Resonance from Compound Asymmetry

The roll and angle-of-attack behavior during spinup and resonance can be obtained from an approximate quasi-steady solution of Eqs. (27), similar to that used in treating the high-altitude resonance problem. It is convenient to treat the spinup and resonance phases separately.

#### B. SPINUP TO RESONANCE

The mechanism for spinup to low altitude resonance with the presence of a body-fixed trim is similar to the roll acceleration during high altitude resonance due to a single mass asymmetry. In both cases, the roll torque is the normal force from angle of attack acting on the center-of-mass asymmetry as its moment arm. During high altitude resonance, the angle of attack is the residual from the exoatmospheric reentry value, which has not completely converged, whereas during spinup, the angle of attack is induced by the aerodynamic trim. In order that the roll torque be sustained, the vehicle must have a preferential roll orientation relative to the wind, which is the lunar motion condition  $\dot{\phi} = 0$ .

Again, if we consider only the quasi-steady vehicle motion by neglecting the angle-of-attack oscillations exemplified by  $\dot{\theta}$  and  $\ddot{\theta}$ , and make the small angle approximations as before, the lunar motion condition requires that  $\dot{\psi} \approx \ddot{p}$ , and we can write Eqs. (27) as

$$\begin{aligned} \tau \cos(\bar{\phi} + \phi_0) - \lambda \bar{\theta} + \delta \cos \bar{\phi} &= -\bar{p}^2(1 - \mu) \bar{\theta} \\ \tau \sin(\bar{\phi} + \phi_0) - \nu \bar{p} \bar{\theta} &= \bar{\theta} \dot{p} \\ -\sigma \sin \phi_0 - \kappa \bar{\theta} \sin \bar{\phi} &= \dot{p} \end{aligned} \quad (29)$$

We can solve these equations approximately for the quasi-steady values  $\bar{p}$  and  $\bar{\theta}$ , for the case  $\phi_0 = \pm 90$  deg, by neglecting the terms  $\delta$  and  $\sigma$ , which are, in general, small relative to other terms in the respective equations. It is shown in Section IV-C, by a comparison of the results so obtained with a numerical integration of Eqs. (27), that this is a good assumption for typical

vehicle and trajectory parameters. Using the definition of  $p_c$ , Eq. (13), we can then reduce Eqs. (29) to

$$\left. \begin{aligned} -\tau \sin \bar{\phi} &= \lambda \left[ 1 - (\bar{p}/p_c)^2 \right] \dot{\theta} \\ \tau \cos \bar{\phi} &= \dot{\theta} \dot{p} \\ -\kappa \dot{\theta} \sin \bar{\phi} &= \dot{p} \end{aligned} \right\} \quad (30)$$

By eliminating  $\dot{\theta}$  and  $\bar{\phi}$  between these equations using the identity  $\sin^2 \bar{\phi} + \cos^2 \bar{\phi} = 1$ , we obtain the nonlinear differential equation for  $\dot{p}$

$$\dot{p} + \frac{\dot{p}^3}{\lambda^2 \left[ 1 - (\bar{p}/p_c)^2 \right]} = \frac{\kappa \tau}{\lambda \left[ 1 - (\bar{p}/p_c)^2 \right]} \quad (31)$$

This is a cubic equation in  $\dot{p}$ , with the second order term missing, which has the exact solution

$$\dot{p} = \left( \frac{\sqrt{3} \kappa \tau}{2x} \right)^{1/2} \left[ x + (1 + x^2)^{1/2} \right]^{1/3} + \left( \frac{\sqrt{3} \kappa \tau}{2x} \right)^{1/2} \left[ x - (1 + x^2)^{1/2} \right]^{1/3}, \quad (32)$$

where

$$x \equiv \frac{3 \sqrt{3} \kappa \tau}{2 \lambda^2 \left[ 1 - (\bar{p}/p_c)^2 \right]^2} \quad (33)$$

For typical vehicle and trajectory parameters,  $x \ll 1$  for  $|\bar{p}/p_c| \lesssim 0.9$ , and Eq. (32) can be expanded in powers of  $x$ , which yields the simple result

$$\dot{p} = \frac{\kappa \tau}{\lambda \left[ 1 - (\bar{p}/p_c)^2 \right]} \quad (34)$$

This indicates that the second term in Eq. (31) can be neglected for  $|\bar{p}/p_c| \lesssim 0.9$ . Since  $p_c^2$  and the coefficients  $\kappa$ ,  $\tau$ , and  $\lambda$  are proportional to the dynamic pressure, which is time dependent, Eq. (34) can be solved for  $\bar{p}$  by a simple numerical integration. Substituting Eq. (34) in Eqs. (30), we find the corresponding angle-of-attack behavior during spinup to be

$$\bar{\theta} = \frac{\pi/\lambda}{1 - (\bar{p}/p_c)^2} \quad (35)$$

This is the same trim amplification factor derived in Section III-C for the subresonance condition with a single mass asymmetry.

### C. LOW ALTITUDE RESONANCE AND ANGLE-OF-ATTACK DIVERGENCE

As the roll rate approaches the critical value Eqs. (34) and (35) no longer apply, and it is essential to include pitch damping  $\nu$  to obtain the roll behavior during resonance. As with the usual resonance phenomenon, the angle of attack exhibits a singularity in the absence of damping as  $\bar{p} \rightarrow p_c$ . Therefore, retaining  $\nu$ , we may write Eqs. (29) with  $\phi_0 = 90$  deg in the form

$$\left. \begin{aligned} \delta \cos \bar{\phi} - \tau \sin \bar{\phi} &= \lambda \left( 1 - \frac{\bar{p}^2}{p_c^2} \right) \\ \tau \cos \bar{\phi} &= (\nu \bar{p} + \dot{\bar{p}}) \bar{\theta} \\ -\sigma - \kappa \bar{\theta} \sin \bar{\phi} &= \frac{\ddot{\bar{p}}}{p} \end{aligned} \right\} \quad (36)$$

As  $\bar{p} \rightarrow p_c$ , the first of Eqs. (36) gives  $\tan \bar{\phi} \rightarrow \delta/\tau = (C_A/C_t)(c/L_t)$ , which, in general, is a very small quantity for trim moments ( $C_t L_t$ ) of sufficient magnitude to spin the vehicle into resonance before impact. Consequently,  $\sin \bar{\phi} \rightarrow \bar{\phi}$  and  $\cos \bar{\phi} \rightarrow 1$ , which gives for  $\bar{\theta}$  during resonance, from the second of Eqs. (36),

$$\bar{\theta} = \frac{\pi}{\nu p_c + \dot{p}_c} \quad (37)$$

Noting that the nonrolling trim angle of attack  $\theta_t$  due to the trim  $\tau$  is  $\tau/\lambda$ , we find that the amplification of the nonrolling trim  $\bar{\theta}/\theta_t$  during resonance is

$$\frac{\bar{\theta}}{\theta_t} = \frac{(1 - \mu)p_c^2}{\nu p_c + \dot{p}_c} \quad (38)$$

When  $\dot{p}_c = 0$ , at peak dynamic pressure, this expression reduces to

$$\frac{\bar{\theta}}{\theta_t} = \left[ \frac{(I - I_x)C_{N_\alpha} l}{\pi \rho} \right]^{1/2} \frac{2\sqrt{2}}{\left[ (C_{N_\alpha} I / md^2) \right] d^3} \quad (39)$$

which, with the exception of  $I$  in place of  $I - I_x$  in the denominator, is the identical expression derived in Ref. 4 using linear theory. The trim amplification factor, Eq. (38), includes the dependence of trim amplification on the slope of the critical roll rate.

The foregoing results provide a simple means for predicting the roll acceleration and angle-of-attack divergence as a function of time (or altitude) after the introduction of an aerodynamic trim. Because of the dependence of the angle of attack during resonance on the slope of the critical roll rate curve and on pitch damping (which is proportional to vehicle velocity), the maximum angle of attack will depend on the altitude at which the vehicle spins into resonance. The approximate angle-of-attack behavior can be constructed from the results of Eqs. (35) and (37).

Figure 9 shows a typical comparison of the quasi-steady approximation with a computer calculation of the roll acceleration to resonance after the introduction of a body-fixed aerodynamic trim 90 deg out of the plane of the center-of-mass asymmetry. Figure 10 shows a similar comparison of the corresponding angle-of-attack divergence.

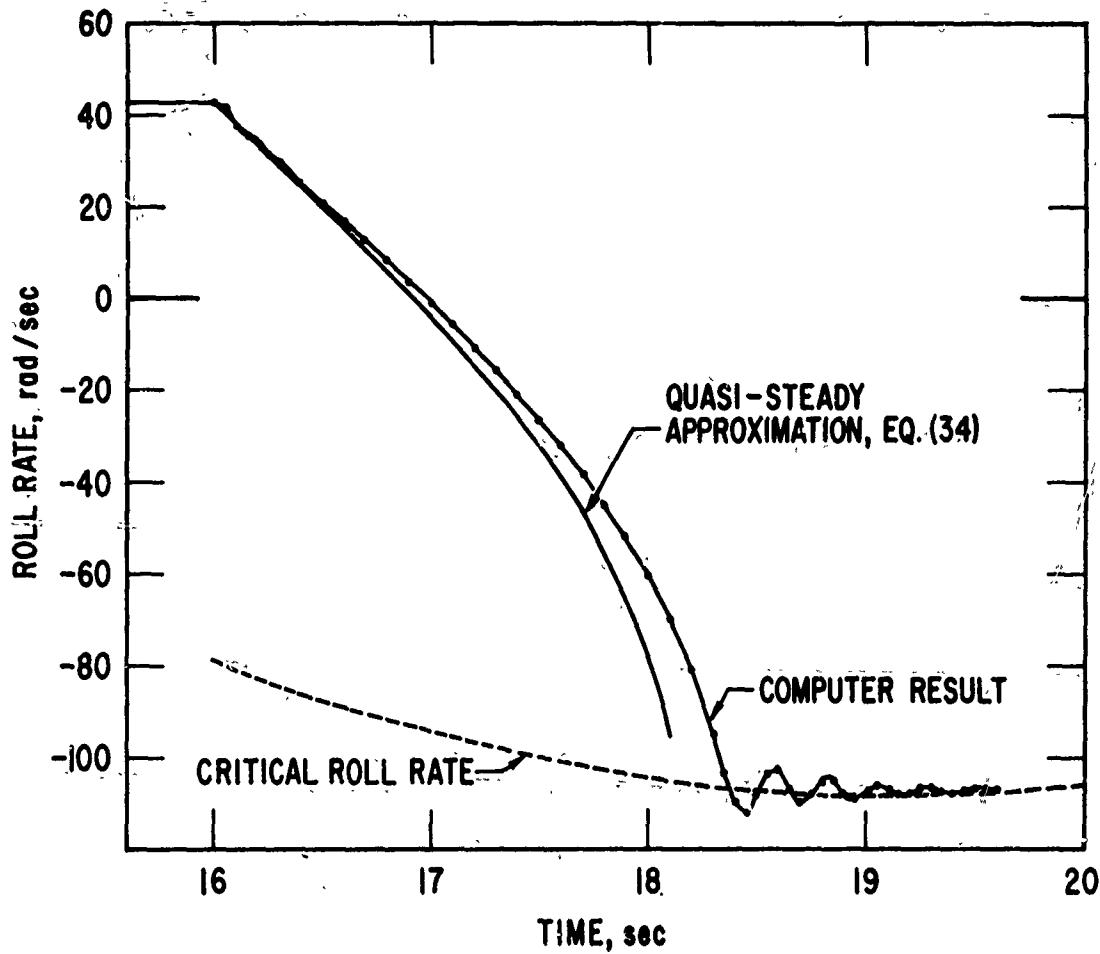
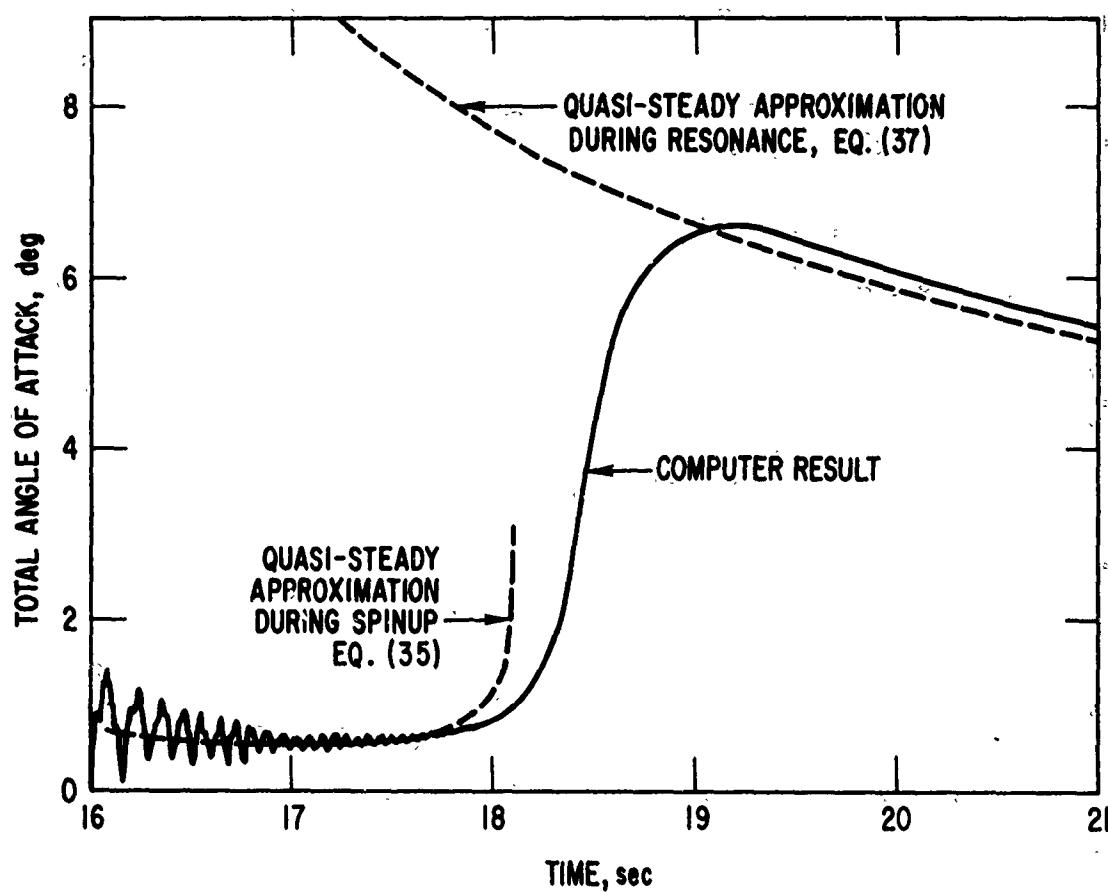


Figure 9. Comparison of Quasi-Steady Approximation with Computer Calculation of Roll Rate Transient after Introduction of Aerodynamic Trim



**Figure 10:** Comparison of Quasi-Steady Approximation with Computer Solution of Angle-of-Attack Divergence after Introduction of Aerodynamic Trim

## V. CONCLUSIONS

It has been demonstrated that reentry vehicle roll resonance can be described by a solution of the moment equations of motion for the quasi-steady or average values of the vehicle orientation and angular rates, without regard for perturbations in the motion about these values. This results in a considerable simplification of the equations of motion.

The analysis has been applied to different regimes of reentry vehicle roll resonance behavior, previously identified through computer solutions of the general equations of motion for particular vehicles and trajectories. The analytical results provide a simple tool for studying, parametrically, reentry vehicle motion for different vehicle configurations and trajectories, and they serve to identify the important parameters and their relative influences on roll resonance phenomena.

It should be possible to extend the analysis to include other aerodynamic forces, such as pure roll torques or forces due to ablation, or mechanically induced forces that might be desirable for control of the vehicle motion.

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## APPENDIX A

### CRITICAL ROLL RATE APPROXIMATION FOR EXPONENTIAL ATMOSPHERE

Using the approach of Allen and Eggers,<sup>2</sup> we can relate the critical roll rate and its derivatives to vehicle and trajectory parameters as follows. It is assumed that atmospheric density  $\rho$  varies exponentially with altitude  $h$  according to

$$\rho = \rho_s e^{-h/H} \quad , \quad (A-1)$$

where  $\rho_s$  and  $H$  are constants. Neglecting acceleration from gravity and assuming a constant axial force coefficient  $C_A$ , we can write the vehicle deceleration from drag in the form

$$-\frac{du}{dt} = \frac{\rho g u^2}{2\beta} \quad (A-2)$$

where  $\beta$  is the ballistic coefficient defined by

$$\beta \equiv \frac{mg}{C_A S} \quad . \quad (A-3)$$

For a constant path angle  $\gamma$ , the vertical descent rate is

$$-\frac{dh}{dt} = u \sin \delta \quad . \quad (A-4)$$

<sup>2</sup>J. H. Allen and A. J. Eggers, Jr., A Study of The Motion and Aerodynamic Heating of Ballistic Missiles Entering the Earth's Atmosphere at High Supersonic Speeds, NACA Report No. 1381 (1958).

Differentiating Eq. (A-1) with respect to time and making use of Eq. (A-4), we find that the rate at which density increases during descent is

$$\frac{dp}{dt} = \frac{\rho u \sin \gamma}{H} \quad . \quad (A-5)$$

Eliminating  $t$  between Eqs. (A-2) and (A-5), we find that the velocity change with density, in terms of the trajectory parameter

$$a \equiv \frac{gH}{\beta \sin \gamma} \quad , \quad (A-6)$$

is

$$\frac{du}{d\rho} = - \frac{ua}{2} \quad , \quad (A-7)$$

which may be integrated to give

$$u = u_0 e^{-(a/2)\rho} \quad , \quad (A-8)$$

where  $u_0$  is the reentry velocity.

The dynamic pressure as a function of density is then

$$q = \frac{1}{2} \rho u^2 = \frac{1}{2} \rho u_0^2 e^{-ap} \quad , \quad (A-9)$$

which has a maximum value

$$q_* = \frac{u_0^2}{2ae} \quad , \quad (A-10)$$

at an altitude

$$h_* = H \ln \rho_s / \rho_* \quad (A-11)$$

where the density is

$$\rho_* = \frac{1}{a} \quad (A-12)$$

and the velocity is

$$u_* = u_o / \sqrt{e} \quad (A-13)$$

The critical roll rate,  $p_c$ , defined by

$$p_c = \left( \frac{C_{N_\alpha} q S I}{I - I_x} \right)^{1/2} \quad (A-14)$$

may now be expressed as a function of atmospheric density and trajectory parameters by substituting for  $q$  from Eq. (A-9), which gives

$$p_c = \left( \frac{C_{N_\alpha} S I}{I - I_x} \right)^{1/2} \frac{u_o}{\sqrt{2}} \rho^{1/2} e^{-(a/2)\rho} \quad (A-15)$$

The time rate of change of  $p_c$ , which is also of interest, is found to be

$$\frac{dp_c}{dt} = \left( \frac{C_{N_\alpha} S I}{I - I_x} \right)^{1/2} \frac{g u_o^2 \rho^{1/2} e^{-ap}}{2 \sqrt{2} \beta} (\rho_* - \rho) \quad (A-16)$$

The atmospheric density is related to altitude through Eq. (A-1) and may be expressed as a function of time through the expression

$$\frac{dp}{dt} = \frac{\rho u_0}{H} e^{-(a/2)\rho} \sin \gamma , \quad (A-17)$$

which may be integrated to give

$$t - t_0 = \frac{H}{u_0 \sin \gamma} \left[ \ln \frac{\rho}{\rho_0} + \frac{(a/2)(\rho - \rho_0)}{1 \cdot 1!} + \frac{(a/2)^2 (\rho^2 - \rho_0^2)}{2 \cdot 2!} + \dots \right] . \quad (A-18)$$

## APPENDIX B

### ANGLE-OF-ATTACK CONVERGENCE DURING ROLL LOCKIN

We can estimate the angle-of-attack convergence through a quasi-steady solution of the equations of motion,

$$\left. \begin{aligned} -\lambda\theta + \delta \cos \phi - v\dot{\theta} &= \ddot{\theta} + \mu p \dot{\psi} \sin \theta - \dot{\psi}^2 \sin \theta \cos \theta \\ \delta \sin \phi - v\dot{\psi} \sin \theta &= \frac{d}{dt}(\dot{\psi} \sin \theta) + \dot{\theta}\dot{\psi} \cos \theta - \mu p \dot{\theta} \\ -\kappa\theta \sin \phi - \epsilon p &= \frac{dp}{dt} \end{aligned} \right\} \quad (B-1)$$

using the exponential atmospheric density approximation discussed in Appendix A. During resonance, the vehicle is in a state of lunar motion and, for small  $\theta$ , the quasi-steady or average values of  $p$  and  $\dot{\psi}$  are equal to the critical roll rate; i. e.,

$$\bar{p} = \bar{\dot{\psi}} = p_c \quad . \quad (B-2)$$

For a vehicle with a large pitch-to-roll inertia ratio,  $\mu$  is small compared with  $\cos \theta$ , and the term  $\mu p \dot{\theta}$  in the second of Eqs. (A-1) can be neglected compared with  $\dot{\theta}\dot{\psi} \cos \theta$ . We can then write this equation, with the condition of Eq. (B-2), in the form

$$\delta \sin \bar{\theta} \sin \bar{\phi} - v p_c \sin^2 \bar{\theta} = \frac{d}{dt} \left( p_c \sin^2 \bar{\theta} \right) \quad . \quad (B-3)$$

If we substitute the expression for  $\sin \phi$  from the third of Eqs. (B-1) into Eq. (B-3), with the small  $\theta$  approximation and the conditions Eq. (B-2), Eq. (B-3) becomes

$$\frac{d}{dt} \left( p_c \sin^2 \bar{\theta} \right) + \nu p_c \sin^2 \bar{\theta} = -\frac{\delta}{\kappa} \left( \epsilon p_c + \frac{dp_c}{dt} \right) . \quad (B-4)$$

This equation can now be integrated for  $p_c \sin^2 \bar{\theta}$  and has the familiar solution.

$$p_c \sin^2 \bar{\theta} = \exp \left[ - \int_{t_1}^t \nu(t') dt' \right] \left\{ \left( p_c \sin^2 \bar{\theta} \right)_{t=t_1} - \frac{\delta}{\kappa} \int_{t_1}^t \exp \left[ \int_{t_1}^{t'} \nu(t'') dt'' \right] \times \left( \epsilon p_c + \frac{dp_c}{dt'} \right) dt' \right\} \quad (B-5)$$

where time  $t = t_1$  corresponds to the point at which the vehicle first locks into resonance, i.e., when the roll rate  $p$  intersects the critical curve  $p_c$ . The angle of attack  $\theta$  at this point can be estimated from the exoatmospheric vehicle motion as follows.

Before reentry, in the absence of external moments and pitch acceleration  $\ddot{\theta}$ , the vehicle will have some exoatmospheric roll and precession motions  $p_o$  and  $\dot{\psi}_o$  at angle of attack  $\theta_o$ , which are gyroscopically coupled according to

$$\dot{\psi}_o = \frac{\mu p_o}{\cos \theta_o} \quad (B-6)$$

Since  $\dot{\psi}_o$  and  $p_o$  will, in general, be unequal, the position of the windward meridian  $\phi$  will rotate continuously about the vehicle, and a steady roll torque cannot persist as it would in the case of lunar motion. However, there can be a net roll torque depending on the phasing of  $\phi$  relative to angle-of-attack oscillations  $\theta$ . Nevertheless, in view of the small magnitude of the aerodynamic

forces at these altitudes, it is assumed that  $p = p_0 = \text{constant}$  until the roll rate intersects the critical curve. If we neglect roll damping  $\epsilon$ , the third of Eqs. (B-1) requires that  $\sin \bar{\phi} = \text{constant}$ . Since, from considerations of the quasi-steady solution of the pitch moment equation discussed in the body of this report, the quasi-steady precession rate  $\dot{\psi}$  will be equal to the natural pitch frequency  $\lambda^{1/2}$ , which is very nearly  $p_c$ , Eq. (B-3) with  $\sin \bar{\phi} = 0$  describes the angle-of-attack convergence prior to high altitude resonance. Integrating this equation from 0 to  $t$ , we obtain

$$p_c \sin^2 \theta = \dot{\psi}_0 \sin^2 \theta_0 \exp \left[ - \int_0^t v(t') dt' \right] , \quad 0 \leq t \leq t_1 . \quad (B-7)$$

We can evaluate the integrals of Eqs. (B-5) and (B-7) using the exponential atmosphere approximations for  $q$  and  $u$  derived in Appendix A. Changing the independent variable from  $t$  to  $\rho$ , using the relation

$$\frac{d\rho}{dt} = \frac{\rho u_0}{H} e^{-(a/2)\rho} \sin \gamma \quad (B-8)$$

derived in Appendix A, and neglecting roll damping in Eq. (B-5), we find that the integrals are

$$\int_0^t v(t') dt' = b \int_{\rho_0}^{\rho} d\rho' = b(\rho - \rho_0) \quad (B-9)$$

$$\begin{aligned} \dot{\phi} &\equiv \frac{\delta}{\kappa} \int_{t_1}^t \exp \left[ \int_{t_1}^{t'} v(t'') dt'' \right] \left( \frac{dp_c}{dt'} \right) dt' = A \int_{\rho_1}^{\rho} e^{\alpha\rho} \left( \frac{1}{a} - \frac{1}{\rho} \right) \frac{dp}{\rho^{1/2}} \\ &= \frac{2Ae^{\alpha\rho}}{a\alpha^{1/2}} \left\{ \frac{b}{\alpha} \left[ D(\sqrt{\alpha\rho}) - e^{-\alpha(\rho-\rho_1)} D(\sqrt{\alpha\rho_1}) \right] - \frac{a\rho^{1/2}}{2\alpha^{1/2}} \left[ 1 - \left( \frac{\rho_1}{\rho} \right)^{1/2} e^{-\alpha(\rho-\rho_1)} \right] \right\} , \end{aligned} \quad (B-10)$$

where

$$\left. \begin{aligned}
 a &\equiv \frac{gH}{\beta \sin \gamma} \\
 b &\equiv \frac{HS}{2 \sin \gamma} \left[ \frac{-C_m d^2 q}{I} + \frac{C_N \alpha}{m} \right] \\
 \alpha &\equiv b - \frac{a}{2} \\
 A &\equiv \frac{\mu C_A}{C_{N_\alpha}} \left( \frac{C_N S l}{I - I_x} \right)^{1/2} \frac{g u_0 H}{2 \sqrt{2} \beta \sin \gamma} e^{-b \rho_1}
 \end{aligned} \right\} \quad (B-11)$$

and  $D(x)$  is Dawson's integral defined by

$$D(x) \equiv e^{-x^2} \int_0^x e^{t^2} dt , \quad (B-12)$$

for which tabular values are available.<sup>3</sup> The angle-of-attack expressions, Eqs. (B-5) and (B-7), can now be written in terms of these results. Of particular interest is the angle of attack when the roll rate first intersects the critical curve at  $t = t_1$ . At this point the critical roll rate will equal the exoatmospheric value  $\dot{\rho}_0$ , which was assumed to remain constant in the interval  $0 \leq t \leq t_1$ . Therefore, from Eq. (B-7) at  $t = t_1$ , using the relation, Eq. (6), we obtain

$$\sin \bar{\theta}_1 = \left( \frac{\mu}{\cos \theta_0} \right)^{1/2} \sin \theta_0 \cdot \exp [ - (b/2)(\rho_1 - \rho_0) ] . \quad (B-13)$$

<sup>3</sup> See, for example, Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Applied Mathematics Series 55, National Bureau of Standards (1964).

For  $t > t_1$ , noting from Eq. (B-13) that

$$(p_c \sin^2 \bar{\theta})_{t=t_1} = \frac{\mu p_o \sin^2 \theta_o}{\cos \theta_o} e^{-b(\rho_1 - \rho_o)} \equiv B , \quad (B-14)$$

We can write Eq. (B-5) in the form

$$\sin \bar{\theta} = \frac{2^{1/4}}{u_o^{1/2}} \left( \frac{I - I_x}{C_N S l} \right)^{1/4} \frac{[B - \frac{1}{4}(\rho)]^{1/2}}{\rho^{1/4}} \exp \left[ \left( \frac{1}{4} \right) (a - 2b)\rho + \left( \frac{b}{2} \right) \rho_1 \right] , \quad t > t_1 , \quad (B-15)$$

which describes the angle-of-attack convergence during resonance.

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## 13. ABSTRACT

Reentry vehicle roll resonance due to small mass and aerodynamic asymmetries is described analytically by a quasi-steady solution of the three-degree-of-freedom moment equations of motion. The quasi-steady analysis neglects oscillations in the vehicle orientation and angular rates about average (quasi-steady) values of these parameters, which change slowly with time relative to the oscillations. The study treats different regimes of reentry vehicle roll resonance that have been identified through computer solutions of the general equations of motion. These include high altitude roll lockin and breakout due to a single mass asymmetry, intermediate and low altitude subresonance with a single asymmetry, and spinup and lockin to low altitude resonance from a compound asymmetry. For each of the resonance regimes, analytical expressions are obtained that describe the quasi-steady vehicle motion, and the results are found to be in good agreement with computer solutions of the equations of motion. The analytical approximations provide a simple tool for predicting reentry vehicle dynamic behavior without requiring costly and time consuming machine computations.

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